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Presented is an elementary approach to areas, columns and other mathematical concepts usually treated in calculus. The approach is based on the idea of average and this concept is utilized throughout the report. In the beginning the average (arithmetic mean) of a set of numbers is considered and two properties of the average which often simplify the arithmetic is noted. Averages are further used to solve a number of important practical problems - to find the work done in stretching a spring, the distance which a body dropped from rest falls in a given time, and the force against a rectangular dam. The volume of solids bounded by two parallel planes is determined by multiplying the distance between the planes by the average cross-sectional area. These volumes can be used to find the force on a dam of triangular or semicircular shape. It is believed that the procedures outlined in this document are sufficiently simple to be taught as early as grade 6. [Not available in hard copy due to marginal legibility of original document]. (RP)

CCSM
Feasibility Study No. 45

Richmond

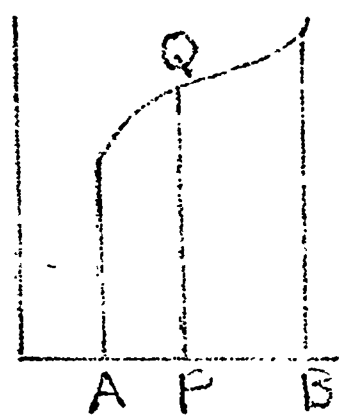
AVERAGES, AREAS AND VOLUMES

1. Introduction

This is a very elementary approach to areas, volumes and other concepts usually treated in the calculus. By introducing the idea of <u>average</u> it is possible to solve these problems in a surprizingly simple way.

We begin by considering the average (arithmetic mean) of a set of numbers and note two properties of the average which often simplify the arithmetic. Then we consider the average of the length AP where P is any point on the line segment from A to B. We show that the average of the lengths AP (denoted by $\underline{AP}$) is $\frac{1}{2}$ AB.

If we define the area under the arc of a curve and above a horizontal base to be the average height $\underline{PQ}$ of points on the curve times the length of the base, we at once obtain the correct results for the area of a right triangle and a trapezoid, and some insight into the area of a circle.



It is also possible to use averages to solve a number of important practical problems: to find the work done in stretching a spring, the distance which a body dropped from rest falls in a given time, and the force against a rectangular dam.

The volumes of solids bounded by two parallel planes may

be determined by multiplying the distance between these planes by the average cross-sectional area. A crucial result is that $\underline{AP}^2 = \frac{1}{3} AB^2$ where $\underline{AP}^2$ is the average of the square of AP for all

points from A to B. This result is easily obtained by considering the volume of a certain square pyramid and is verified by an arithmetical computation. The average $\underline{AP}^2$ immediately gives the well-known results for the volume of a cone and a sphere. Various other volumes may be obtained from this average.

It is possible to use these volumes to find the force on a dam of triangular or semicircular shape.

It is believed that the method of averages is sufficiently simple to be taught quite early, hopefully in the 6th grade. The method is intuitively reasonable and immediately leads to results of great usefulness. It therefore should impress the student with the power of mathematics. Moreover, it connects nicely with the introduction of statistical ideas at an early stage.

It is presupposed that the student is acquainted with the fact that the area of a rectangle is the length of the base times the height and the volume of a cube is the cube of the length of a side. He should also know that the corresponding sides of similar triangles are proportional

and he should be able to work with proportions in simple ways. For the volume of the sphere and some other figures, the Pythagorean Theorem must be used.

Throughout we use the conventional symbol $\overline{AP}$ for the line segment joining the points A and P and AP for its length. The new symbol $\underline{AP}$ is introduced for the average of AP over an interval $\overline{AB}$.

Certain parts of this program may be omitted. The last section on the area below a parabola is perhaps a luxury although it is very simple and quite surprizing. The force problems in section 8 may be a trifle ambitious. The calculations in section 5 need not be carried as far as we have done.

2. The Average of a Set of Numbers

Suppose that a student obtains the following grades in his mathematics course: 80, 75, 100, 95. What single number best represents his knowledge of the subject? How would he compare with a second student whose grades in the same examination were 85, 70, 95, 100?

The first question may be answered by taking the *average* of the four numbers as follows:

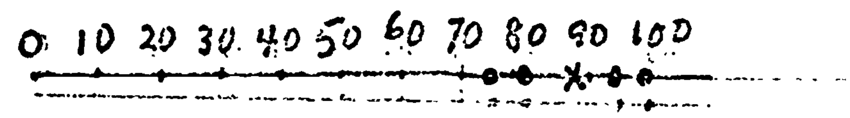
$$\frac{80 + 75 + 100 + 95}{4} = \frac{350}{4} = 87\frac{1}{2}.$$

The second student has the same average since

$$\frac{85 + 70 + 95 + 100}{4} = \frac{350}{4} = 87\frac{1}{2}.$$

As we see, the method of finding the average of a set of numbers is very simple. We add the numbers and then divide the sum by the number of items which are to be averaged.

The result of averaging a set of numbers may be shown on the number line.



For the first student the result would look like this where the dots show the numbers to be averaged and the cross shows their average.

There are several ways in which we can often reduce the amount of arithmetic to be performed. For example for the list of numbers

75, 80, 95, 100

we can subtract the smallest from each of them and obtain the new list

0, 5, 20, 25

whose average is $\frac{0 + 5 + 20 + 25}{4} = \frac{50}{4} = 12\frac{1}{2}$.
If we add $12\frac{1}{2}$ to 75 we find $87\frac{1}{2}$ as before. On the number line this means that we do not change the position of the cross if we measure from the left-most point rather than from 0. Indeed, in drawing so much of the number line as we did before, we were wasting paper.

Instead of measuring from the smallest value we can measure from any convenient place. We may guess the average and measure from the position of our guess. With our example, 85 appears to be a good guess. Let us measure from 85 so that 85 is subtracted from each of the numbers

$$75, 80, 95, 100 .$$

The new list

$$-10, -5, 10, 15$$

of course contains some negative numbers. This makes the addition very simple. The new average is

$$\frac{-10 \quad -5 \quad +10 \quad +15}{4} = \frac{10}{4} = 2\frac{1}{2} .$$

If we add $2\frac{1}{2}$ to 85 we obtain $87\frac{1}{2}$ once again.

There is another idea which sometimes simplifies the arithmetic. It may be that all of the numbers in a set to be averaged have a common factor. Take for example the set

$$100, 300, 500, 700 .$$

The average of course is

$$\frac{100 + 300 + 500 + 700}{4} = \frac{1600}{4} = 400 .$$

But it would be simpler to take out the common factor 100, find

$$\frac{1 + 3 + 5 + 7}{4} = \frac{16}{4} = 4$$

and multiply the result by 100.

Do you see what this new method for simplifying the arithmetic means when applied to the number line? It means that the position of the cross does not change if we choose a new unit for measuring length. In the example the new unit was 100 old units.

The methods that we have used to simplify the arithmetic of finding an average are based on two principles:

(1) The effect of adding a given number to each of a set of numbers is to increase their average by the given number;

(2) The effect of multiplying each of a set of numbers by a given number is to multiply their average by the given number.

Exercises

1. Eight boys have the following heights:

5 ft. 8 in., 5 ft. 9 in., 5 ft. 10 in.
5 ft. 11 in., 6 ft. 0 in., 6 ft. 1 in.,
6 ft. 2 in., 6 ft. 3 in.

Find their average height (Hint: Measure from 6 ft.)

2. The number of passengers who landed in Nairobi airport for each quarter of 1963 were

1st quarter	32,100
2nd quarter	31,200
3rd quarter	38,500
4th quarter	34,500

Find the average number of passengers who landed per quarter.

3. Average the numbers

   a) 1, 2
   b) 1, 2, 3
   c) 1, 2, 3, 4
   d) 1, 2, 3, 4, 5
   e) 1, 2, 3, 4, 5, 6

Can you think of a general rule so that you could tell for example without doing a long addition what is the average of all of the whole numbers from 1 to 10 (inclusive)? 1 to 100?

Hint: Write each of your results as a certain number of halves.

4. Average the numbers

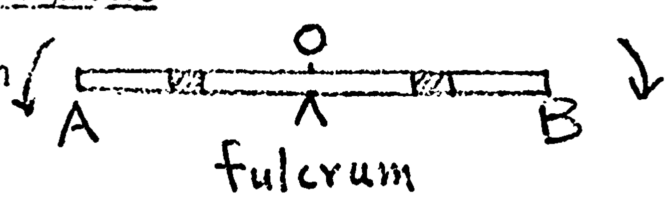
   a) 1, 3
   b) 1, 3, 5
   c) 1, 3, 5, 7
   d) 1, 3, 5, 7, 9

Can you discover a general rule for the result?

5. Find the average of .1, .2, .3, .4, .5, .6, .7, .8, .9, and 1.0 as quickly as possible.

3. <u>The Average Length of a Line Segment</u>

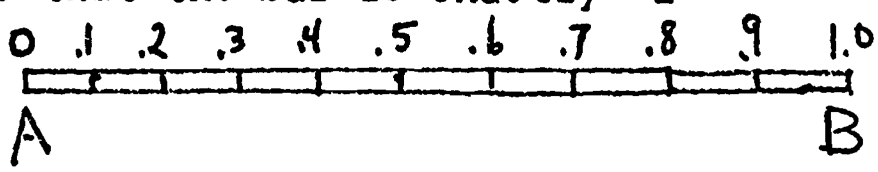
Everyone knows that a uniform bar will balance at its midpoint. The parts of the bar on the right of the fulcrum tend to turn the bar clockwise. Those parts on the left of the fulcrum tend to turn the bar in the opposite (anti-clockwise) direction. But, to



every chunk of material on the right there corresponds a chunk of the same size and weight at the same distance on the left. If we place a number line along the bar with its 0 at the fulcrum, points on the right of 0 will correspond to positive numbers and those on the left to negative numbers. We may think of 0 as the average of the numbers positive and negative that go with <u>all</u> of the points of the bar from A to B.

If however we measure distance from the left end of the bar the average of the lengths AP for all points P on the bar from one end to the other (from A to B) should correspond to the previous point 0. Let us write the average of AP over the bar as $\underline{AP}$. We see therefore that it must be true that $\underline{AP} = \frac{1}{2}$ AB.

This intuitive argument can be supported by a few calculations which relate the present averages to those that we worked out in section 2. For simplicity we shall choose our unit of length so that the bar is exactly 1 unit long. Let us imagine that the bar is marked at the points .1, .2, .3, .4, .5, .6, .7, .8, .9.



If all of the material in the section from 0 to .1 were pushed out to the distance .1 and similarly if the material in each of the other subdivisions were pushed out to its further end, it is clear that the effect would be to increase the average distance of P from A. But

if this were done the average would be

$$\frac{.1 + .2 + .3 + .4 + .5 + .6 + .7 + .8 + .9 + 1.0}{10} = .55 .$$

Do you remember how to do this quickly? We can conclude that $\underline{AP} < .55$ where "<" is read "is less than."

On the other hand, let us imagine that the material in each subdivision of length .1 is pushed to the left-most point of its interval. This would surely decrease the average distance of P from A. But then we would have the average

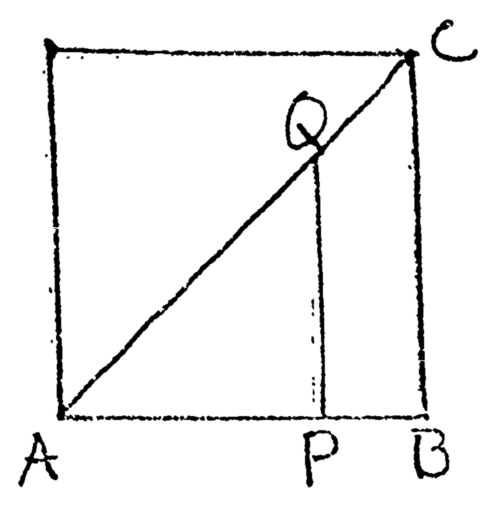
$$\frac{0 + .1 + .2 + .3 + .4 + .5 + .6 + .7 + .8 + .9}{10} = .45 .$$

Therefore $\underline{AP} > .45$ where ">" means " is greater than." So surely $\underline{AP}$ is between .45 and .55.

If we use the same idea with intervals of length .01 it turns out that $\underline{AP}$ is between .495 and .505. Can you show this in a simple way? If we go on and use intervals of length .0001 we shall find that $\underline{AP}$ lies between .4995 and .5005. We could go even further but surely by now it is clear that $\underline{AP} = \frac{1}{2}$ is the right answer.

We have taken AB to be 1. If we do not do this, our result will read

$$\underline{AP} = \frac{1}{2} AB.$$



Let ABC be a right triangle with equal legs AB and BC as shown in the figure. Let P be any point on its base and let Q

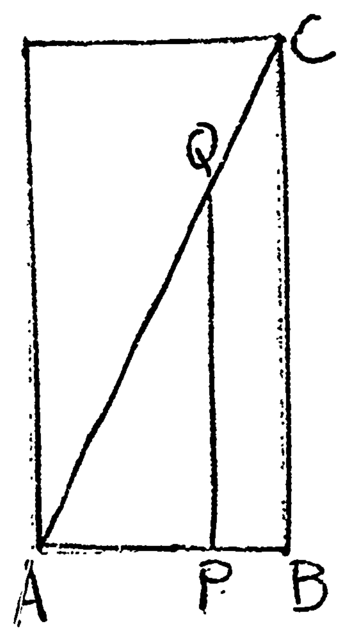
be the point on $\overline{AC}$ that is directly above P. The length PQ is the height of the point Q above the base. What is the average value of PQ for all points P of the base? Since $PQ = AP$, $\underline{PQ} = \underline{AP} = \frac{1}{2} AB$.

It is reasonable to think of the area of triangle ABC as the length of the base times the average height of Q.

That is

$$\text{Area} = AB \times \frac{1}{2} AB = \frac{1}{2} AB^2.$$

Is this correct? It surely is because the area of a square with side AB is AB^2 and the triangle ABC is exactly half of such a square.



We can carry this idea further. Let ABC be a right triangle where this time we do not require that $BC = AB$. It might be for example that $BC = 2AB$. If we say that the area of triangle ABC is the base AB times the average value of PQ how do we come out? If $BC = 2AB$ $PQ = 2AP$. Then

$$\text{Area} = AB \times \underline{PQ} = AB \times \underline{2AP}$$

Now the average of 2AP is twice the average of AP. So

$$\underline{2AP} = 2\ \frac{1}{2}\ AB = \frac{1}{2}\ BC$$

and

$$\text{Area} = \frac{1}{2} AB \times BC.$$

This is correct. Triangle ABC is one half of the rectangle ABCD whose area is $AB \times BC$ (Length of base

x height).

If instead of taking $BC = 2\ AB$ we assume that $BC = c\ AB$ where c is any positive number the argument would work just as well. We have

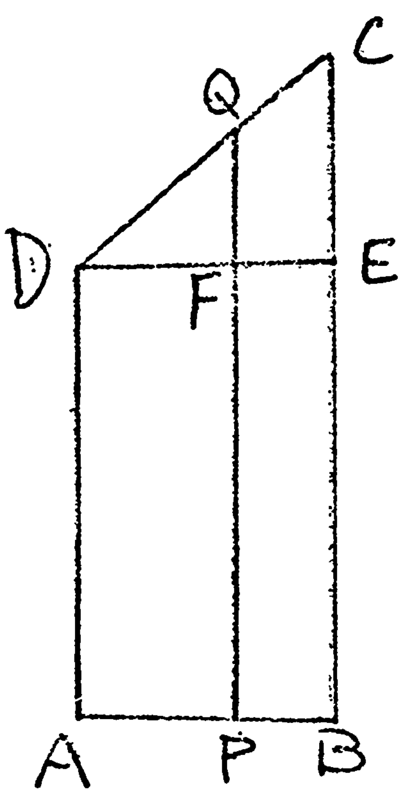
$$PQ = c\ AP$$

$$\underline{PQ} = c\ \underline{AP} = c \times \frac{1}{2} AB = \frac{1}{2}(c \times AB) = \frac{1}{2} BC$$

$$\text{area} = AB \times \frac{1}{2} BC$$

which again is one half the area of the rectangle ABCD. We have simply used our second principle for simplifying averages in carrying out this argument.

To find the area of the trapezoid in the figure we may start with



Area = $AB \times \underline{PQ}$.

Since $PQ = PF + FQ = AD + FQ$

$\underline{PQ} = AD + \underline{FQ}$ (Why?)

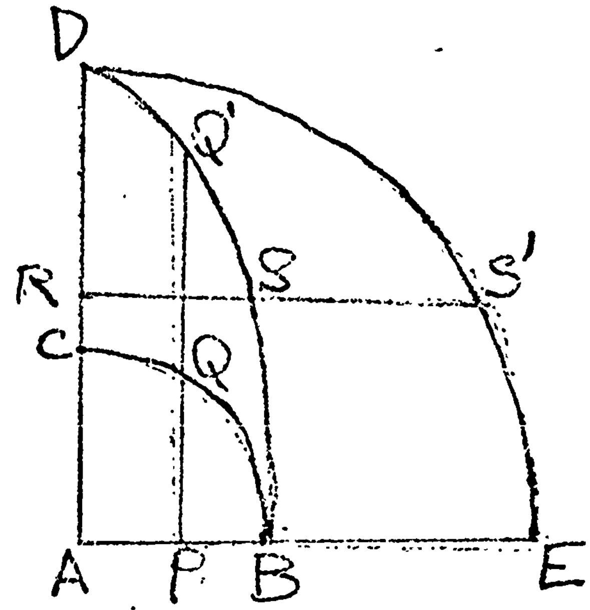
But $\underline{FQ} = \frac{1}{2} EC$ as you should be able to show. So the required area is $AB \times (AD + \frac{1}{2} EC)$.

Show that this is the same as

$$AB \times \frac{AD + BC}{2}$$

and express this result in words.

The area of a circle is more difficult. Let us consider ABC, one quarter of the region bounded by a circle of radius 1.



(In the figure, $AB = 1$). If we knew the area a of ABC, we would have no difficulty with the area of a circle of any other radius r.

We can see this as follows.

Since $AB = 1$, the average $\underline{PQ} = a$.

Now let each PQ be multiplied by r ($PQ' = rPQ$). Then $\underline{PQ'} = r\underline{PQ} = ra$ is the area of the figure ABD. This area may be written as $\underline{RS} \times AD = \underline{RS} \times r$.

Since

$$ar = \underline{RS} \times r$$

$$\underline{RS} = a.$$

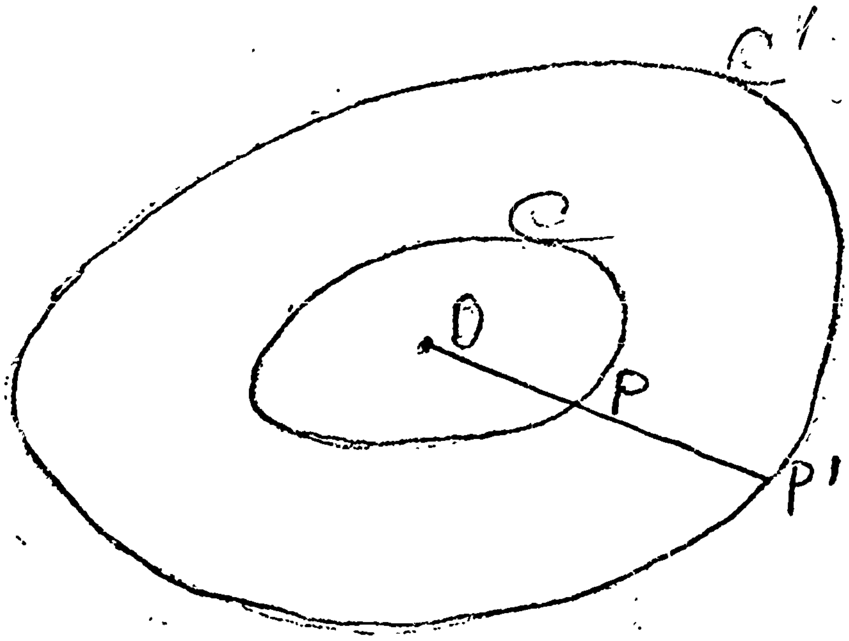
Let us now multiply each RS by r ($RS' = rRS$). The result is the quarter circle ADE of radius r. What is its area?

$$\text{Area} = \underline{RS'} \times AD = r\underline{RS} \times r = r \times a \times r = ar^2$$

Hence the area within a circle of radius r is $4\ ar^2$

It is customary to use the letter π to represent the area $4a$ of a circle of radius 1 so that the area of a circle of radius r is πr^2. It turns out that π is approximately equal to 3.1416 but we shall not show how this number is obtained.

Let C be a closed curve and O a point in its interior. Let C' be a curve obtained by connecting all the points P' which are found by multiplying OP by a constant c. An argument like that used for the circle shows that the area enclosed by C' is c^2 times the area enclosed by C.

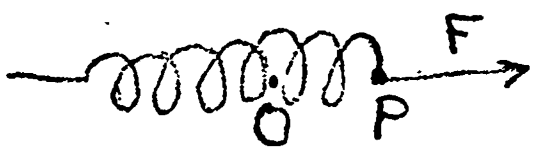


4. Applications

Averages may be used for many other purposes than that of finding areas. We give some examples.

A. Robert Hooke discovered that the force exerted by a spring is proportional to the amount by which the spring is stretched.

That is, if OP is the amount of stretch and F is the force, then



$F = c \times OP$ for some constant c. Let F be measured in pounds and OP in feet. c will be the number of lbs per ft of stretch. It will depend on the stiffness of the spring.

When a body is moved through a given distance against a constant force we say that the amount of work done (or

energy expended) is the force times the distance. With our units, the work will be measured in ft-lbs. The question arises: How much work is done in stretching a spring an amount OA? What shall we do if the force changes as we move? It is natural to say that

Work = average force x distance.

In the case of the spring where $F = c \times OP$,

$$\underline{F} = c \times \underline{OP} = c \times \frac{1}{2}\ OA$$

and the work is $c \times \frac{1}{2} OA \times OA$

or

$$\text{Work} = \frac{1}{2} c \times OA^2$$

This turns out to make physical sense.

B. Galileo assumed that when a body is dropped from rest its speed (or velocity) increases uniformly with time so that

speed = c x time

where c is the constant increase of speed per second. We now know that Galileo was right and that c is about 32 ft/sec increase per second. The question that Galileo asked was this. If we assume that the speed does increase in this way, how far will a body fall in a given time? If speed does not change, the distance is the speed multiplied by the time. Galileo replaced this by

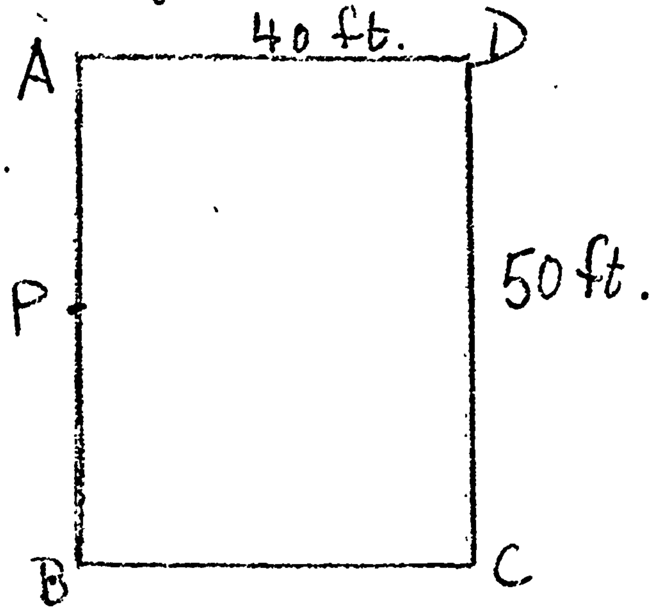
distance = average speed x time.

If we use the length OP on a scale to represent the time in seconds and if OA is the total time which elapses,

$$\text{distance} = \underline{32\ OP} \times OA$$
$$= 32 \times \tfrac{1}{2}\ OA \times OA$$
$$= 16 \times OA^2\ .$$

For example the distance which a body falls from rest in 1 second is about 16 ft, in 2 seconds about 64 ft, and so on. Results of this sort are verified by experiment so that Galileo's assumption has been justified.

C. Let ABCD be a rectangular dam in a vertical plane with the dimensions shown. We wish to find the force exerted on the dam by the water which presses against it. It is known that the pressure at depth AP ft. is about



$$62.5 \times AP \ \text{lbs. per sq. ft.}$$

The average pressure over the face of the dam is

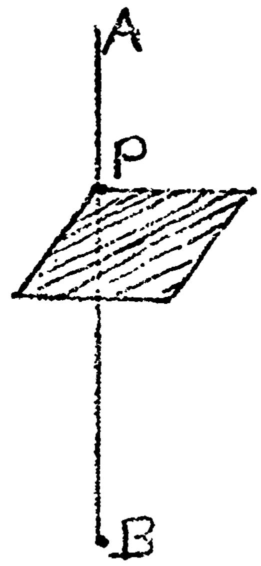
$$62.5 \times \tfrac{1}{2}\ AB = 62.5 \times 25 = 1562.5 \ \text{lbs/sq. ft.}$$

There are of course $40 \times 50 = 2000$ sq. ft. on the face of the dam so that the total force is 1562.5×2000 lbs. or 1562.5 tons.

5. Volumes and Averages

Let P be any point on a line segment $\overline{AB}$. To make things simpler we shall

choose a unit of length so that $AB = 1$. Now imagine that at every point P we draw a square region perpendicular to $\overline{AB}$ with each side equal to AP and so that one corner of the square is at P. If all possible such squares are properly drawn we shall fill up a pyramid $A - BCDE$ with height one unit and a square base one unit on a side.



The square with a corner at P is a cross-section of this pyramid. Its area is AP^2. What is the average $\underline{AP^2}$ of all of these cross-sectional areas?

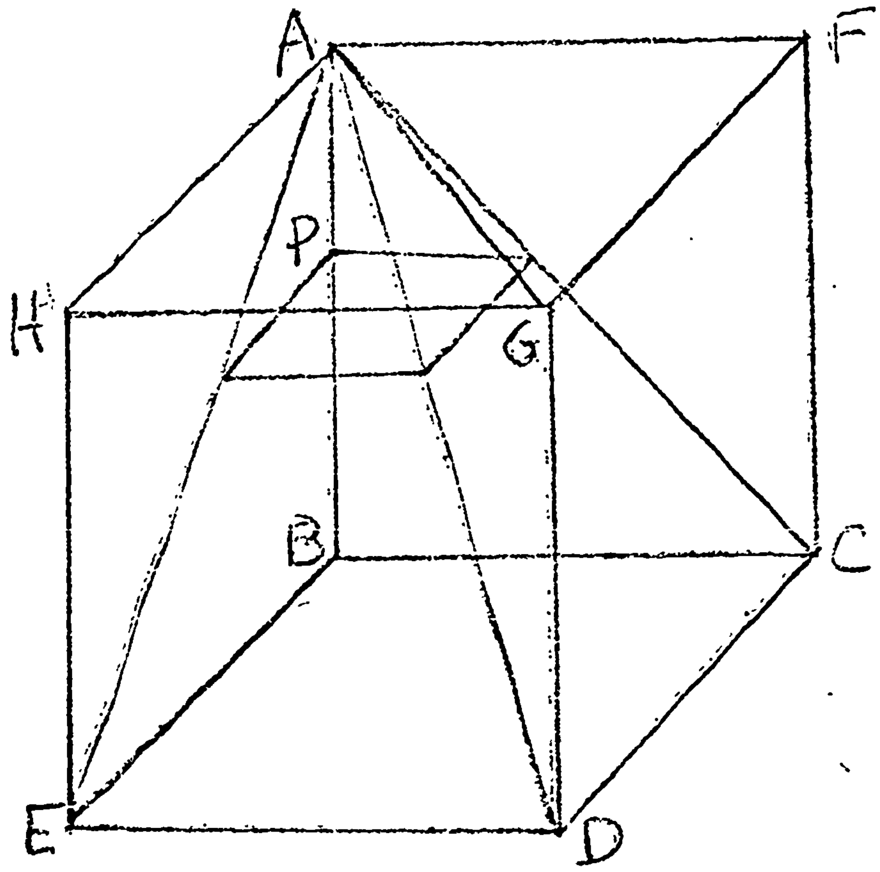
This is very easy to answer. Let us agree that the volume of the pyramid is the average cross-sectional area times the height. Then

$$\text{Volume} = \underline{AP^2} \times AB = \underline{AP^2}$$

since $AB = 1$. But the volume of the pyramid is $\frac{1}{3}$ the volume of a cube one unit on a side. That is, the volume is $\frac{1}{3}$ of a cubic unit and therefore

$$\underline{AP^2} = \frac{1}{3}.$$

How do we know that the volume of the pyramid $A - BCDE$ is $\frac{1}{3}$ of the volume of a unit cube? By looking at this picture.



Do you see our pyramid A -BCDE? Now do you see the pyramid A--FCDG with square base FCDG one unit on a side and with the unit line segment perpendicular to it at the corner F?

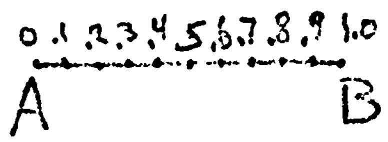
Finally do you see the pyramid A -HEDG with square base HEDG one unit on a side and with the unit segment $\overline{AH}$ perpendicular to the base at the corner H?

These three pyramids are exactly alike. We say that they are congruent. Since together they make up the unit cube of volume 1, each pyramid must have the volume 1/3.

(We suggest the construction of a model to show this)

We see therefore that $\underline{AP^2} = \frac{1}{3}$.

Let us check this result with a little arithmetic. Suppose that we mark $\overline{AB}$ at the points .1, .2, .3, .4, .5, .6, .7, .8, .9 and 1, and take the squares of each of these numbers.



.01, .04, .09, .16, .25, .36, .49, .64, .81, 1.00

If we imagine that the areas of all the cross-sections with P between A and .1 are replaced by $(.1)^2 = .01$ and similarly that in each interval of length .1 all cross-sections are pushed out to the further end we shall surely

increase the average cross-sectional area. What result shall we get? If you average the numbers

.01, .04, .09, .16, .25, .36, .49, .64, .81, 1.00

you will find .385, a result which is certainly too large for $\underline{AP}^2$.

If we replace the cross-sectional areas in each .1 unit interval by the smallest cross-sectional area on this interval we have the numbers

0, .01, .04, .09, .16, .25, .36, .49, .64, .81

whose average is .285. Can you think of an easy way to find this result from the previous one? The result .285 is too small to be $\underline{AP}^2$.

Thus our required average $\underline{AP}^2$ is surely between .285 and .385.

We could of course find a better estimate for $\underline{AP}^2$ if we subdivided $\overline{AB}$ into 100 equal parts and followed the same procedure as before. However the arithmetic looks rather terrifying. Let us see if we cannot work out a simple scheme.

At the end of section 2, we found a short way to average any number of consecutive whole numbers beginning with 1. We found that it was sufficient to add the first and last and divide by 2.

For example

$$\frac{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10}{10} = \frac{1 + 10}{2} = \frac{11}{\cancel{5}\ 2}$$

What is $\dfrac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2}{10}$?

It is certainly not $\dfrac{1^2 + 10^2}{\cancel{10}\,2} = \cancel{10.1}\ 50.5$

Let us go back to our simple averages

$$\frac{1 + 2}{2} = \frac{3}{2}$$

$$\frac{1 + 2 + 3}{3} = 2$$

$$\frac{1 + 2 + 3 + 4}{4} = \frac{5}{2}$$

$$\frac{1 + 2 + 3 + 4 + 5}{5} = 3$$

Let us work with the squares in the same way.

$$\frac{1^2 + 2^2}{2} = \frac{5}{2}$$

$$\frac{1^2 + 2^2 + 3^2}{3} = \frac{14}{3}$$

$$\frac{1^2 + 2^2 + 3^2 + 4^2}{4} = \frac{15}{2}$$

$$\frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5} = 11$$

It is not so easy to see any system in the results. But we do see that the averages of the squares increase faster than the simple averages. Let us make a table and compare the two averages by dividing.

Number of Items	Average of Squares	Average of Numbers	$\frac{\text{Average of Squares}}{\text{Average of Numbers}}$
2	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{5}{3}$
3	$\frac{14}{3}$	2	$\frac{7}{3}$
4	$\frac{15}{2}$	$\frac{5}{2}$	3 $\left(\frac{9}{3}\right)$
5	11	3	$\frac{11}{13}$

If we write 3 as $\frac{9}{3}$ we begin to see the system. To find the average of the squares from the simple average of the numbers we should multiply by $\frac{2n + 1}{3}$ where n is the number of items to be averaged.

Let us try this out. Suppose $n = 10$.

The average

$$\frac{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10}{10} \text{ is } \frac{11}{2}$$

The average

$$\frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2}{10}$$

should be $\frac{21}{3} = 7$ times as great, that is, $\frac{11}{2} \times 7 = \frac{77}{2} = 38.5$.

You will find that this is correct. In fact we found earlier by the long way that

$$\frac{.1^2 + .2^2 + .3^2 + .4^2 + .5^2 + .6^2 + .7^2 + .8^2 + .9^2 + 1.0^2}{10}$$

is .385 which is just what we would expect.

Now let us jump to the average of

$$.01^2,\ .02^2,\ \cdots\ ,\ 1.00^2$$

which is $\frac{1}{10,000}$ of $\frac{1^2 + 2^2 + 3^2 + \cdots + 100^2}{100}$.

The simple average of the numbers from 1 to 100 is $\frac{101}{2}$. The average of the squares of these numbers is $\frac{201}{3} = 67$ times as great, that is,

$$\frac{101}{2} \times 67 = \frac{6767}{2} = 3383.5.$$

If we divide by 10,000 we have the result that we wanted .33835. This is too large to be the true average $\underline{AP^2}$

Can you show with almost no work that .32835 is too small?

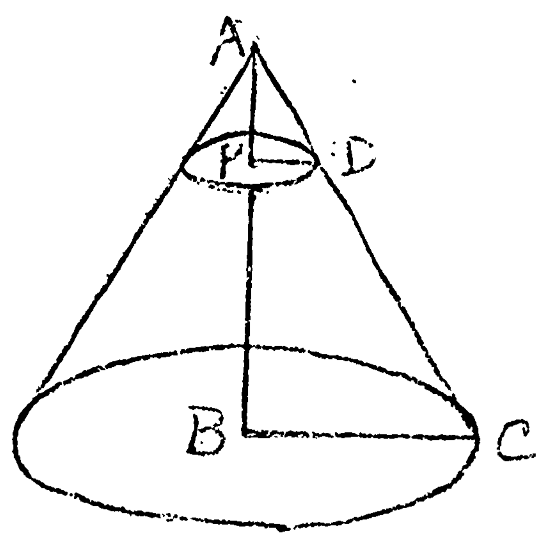
As you see, our result $\underline{AP^2} = \frac{1}{3}$ is borne out by our arithmetic, We take $\underline{AP^2} = \frac{1}{3}$ to be exactly true. If AB is not 1, the argument from the volume of the pyramid gives

$$\underline{AP^2} = \frac{1}{3}\ AB^2$$

since this time the volume of the cube is AB^3.

6. <u>Cones and Spheres</u>

<u>The Volume of a Cone.</u>



Let C be a circle with center at B and A a point on the perpendicular to the plane of the circle through B.

If we join A to each of the points of the circle we form the surface of a cone. What is the volume enclosed by the base and the conical surface? This is easy to answer.

If P is any point on $\overline{AB}$, the cross-section at P is a circle whose area is π times the square of its radius, PD^2. The simplest case is that in which the height AB of the cone is equal to BC, the radius of its base. Then $PD^2 = AP^2$.

$$\pi\ \underline{PD^2} = \pi\ \underline{AP^2} = \pi \times \frac{1}{3} AB^2.$$

To find the volume we multiply by the height AB to obtain

$$\text{Volume} = \pi \times \frac{1}{3} AB^3.$$

More generally, suppose $BC = c\ AB$ where c is any positive number. Then $PD = c\ AP$ and the area of the cross-section is $\pi\ PD^2 = \pi\ c^2\ AP^2$.

The average is

$$\pi\ c^2\ \underline{AP^2} = \pi c^2 \times \frac{1}{3} AB^2$$

and the volume is $\pi\ c^2 \times \frac{1}{3} AB^3$.

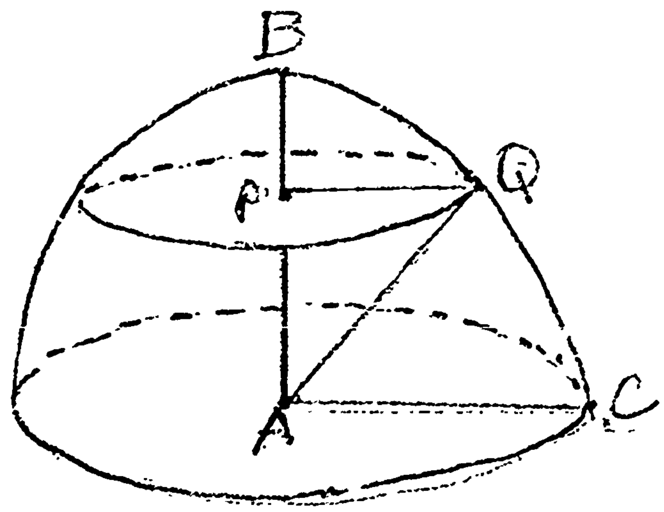
Since $c^2 = \dfrac{BC^2}{AB^2}$,

$$\text{Volume} = \frac{1}{3}\ \pi\ \frac{BC^2}{AB^2} \times AB^3$$

$$= \frac{1}{3}\ \pi\ BC^2 \times AB.$$

In words, the volume of the cone is one third the product of the area of the base by the height.

The Volume of a Hemisphere



To find the volume of the hemisphere shown we must multiply the height AB by the average cross-sectional area. The section at P is a circle whose area is πPQ^2.

By the Pythagorean Theorem

$$PQ^2 + AP^2 = AQ^2$$

where of course AQ is the radius of the hemisphere which is independent of the position of P. We must average

$$\pi PQ^2 = \pi(AQ^2 - AP^2)$$

The result is $\pi\,(AQ^2 - \underline{AP^2})$

$$= \pi\,(AB^2 - \tfrac{1}{3}AB^2)$$

$$= \frac{2\pi}{3}AB^2.$$

The volume is this average cross-sectional area times AB. Therefore

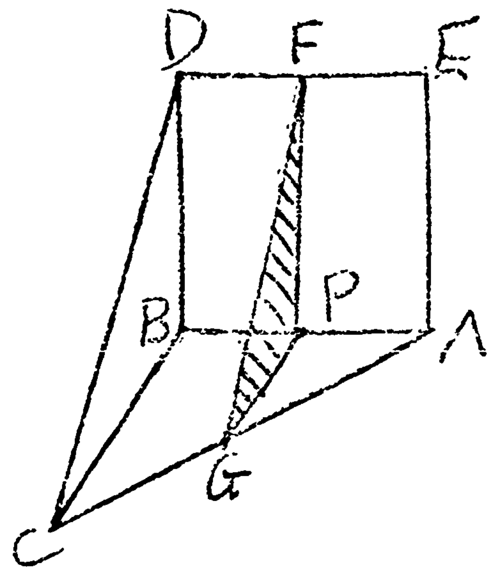
$$\text{Volume} = \frac{2\pi}{3}AB^3.$$

The volume of a whole sphere of radius AB is $\frac{4\pi}{3}AB^3$,

7. Volumes of Other Figures

We can now find the volumes of a number of other interesting figures . We give four examples.

A. Let ABC be a right triangle with AB = BC and ABDE a square with side AB in a plane perpendicular to that of ABC. Form a solid as follows. From each point F on $\overline{DE}$ drop a perpendicular $\overline{FP}$ onto $\overline{AB}$. Now in the plane of ABC draw a perpendicular $\overline{PG}$ to $\overline{AB}$ at P meeting $\overline{AC}$ at G. Connect F and G.



The area of the cross-section FPG is

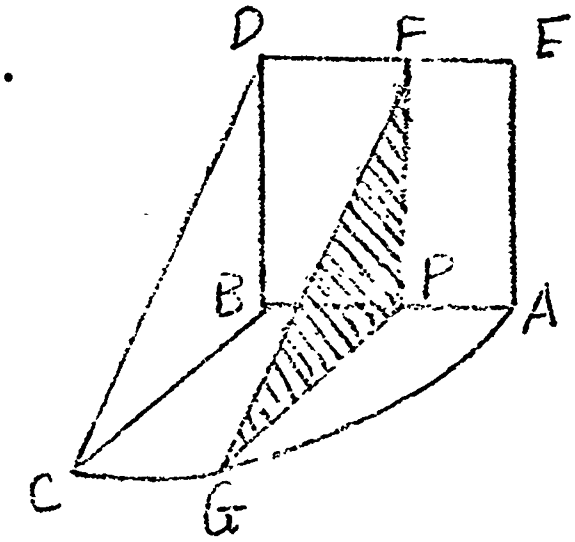
$$\frac{1}{2} FP \times PG = \frac{1}{2} FP \times AP.$$

The average of these areas for all points P from A to B is

$$\frac{1}{2} FP \times \frac{1}{2} AB = \frac{1}{4} AB^2 \text{ since } FP = AB.$$

The volume is $\frac{1}{4} AB^2 \times AB = \frac{1}{4} AB^3$.

B. In the previous figure let the line segment $\overline{AC}$ be replaced by the quarter circle AC. If the figure is drawn as before, what will the volume be?



The area of triangle FPG is $\frac{1}{2} FP \times PG = \frac{1}{2} AB \times PG$. Its average is of course $\frac{1}{2} AB \times \underline{PG}$. But what is $\underline{PG}$?

Since $\underline{PG} \times AB$ is the area of ABC, a quarter circle with

radius AB,

$$\underline{PG} \times AB = \frac{1}{4} \pi AB^2$$

and $$\underline{PG} = \frac{1}{4} \pi AB.$$

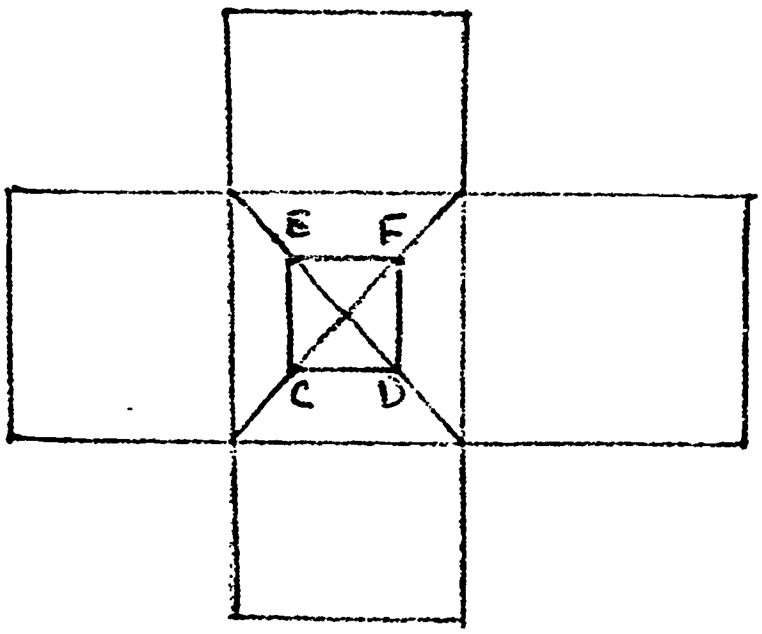
Therefore the volume is

$$AB \times \frac{1}{2} AB \times \underline{PG}$$

$$= AB \times \frac{1}{2} AB \times \frac{1}{4} \pi AB$$

$$= \frac{1}{8} \pi AB^3.$$

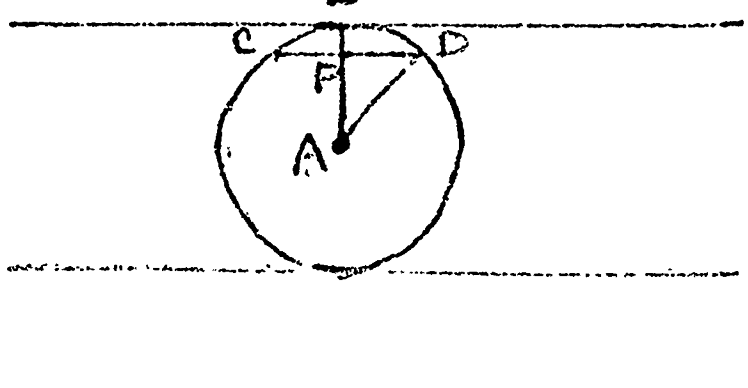
C. Two circular cylinders of equal radius intersect so that their axes cross at right angles. What is the volume of their intersection?



Here is a top view. A horizontal section is a square CDEF with area CD^2. If we look along the axis of one of the cylinders we see the line segment $\overline{CD}$ edge on and notice that $CD = 2\ PD$. Then $CD^2 = 4\ PD^2$

$$= 4(AD^2 - AP^2)$$

$$= 4(AB^2 - AP^2)$$



where we have used the Pythagorean Theorem. The average cross-sectional area over the

upper half of the intersecting cylinders is

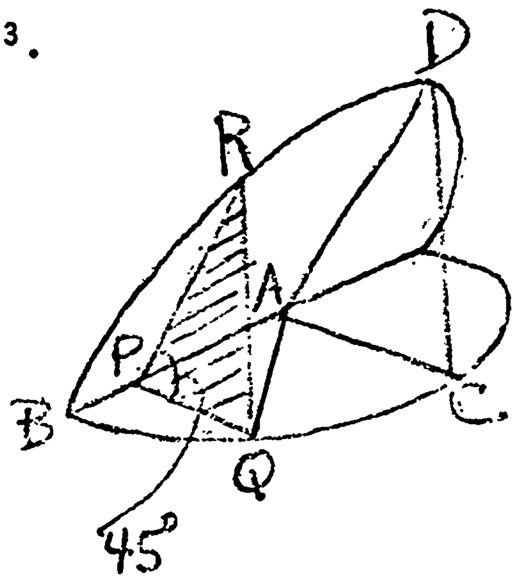
$$\underline{CD}^2 = 4(AB^2 - \underline{AP}^2) = 4(AB^2 - \frac{1}{3} AB^2) = \frac{8}{3} AB^2.$$

One half of the required volume is

$$AB \times \frac{8}{3} AB^2.$$

The volume of the whole figure is $\frac{16}{3} AB^3$.

(A model for this figure is suggested)



D. A wedge is cut from a cylindrical tree so that the edge of the cut lies along a diameter. One plane is horizontal and the other plane makes a $45°$ angle with it. What is the volume?

We shall consider half of this wedge, the part which lies above ABC.

A section PQR perpendicular to $\overline{AB}$ is an isosceles right triangle with area

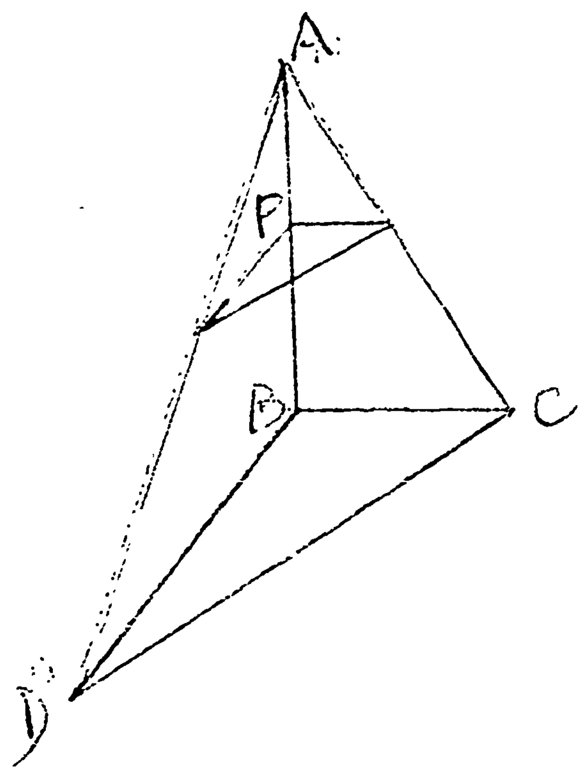
$$\frac{1}{2} PQ \times QR = \frac{1}{2} PQ^2 = \frac{1}{2}(AQ^2 - AP^2)$$
$$= \frac{1}{2}(AB^2 - AP^2).$$

The average for all points P from A to B is

$$\frac{1}{2}(AB^2 - \frac{1}{3} AB^2) = \frac{1}{3} AB^2.$$

The volume of the half-wedge is $AB \times \frac{1}{3} AB^2$ or $\frac{1}{3} AB^3$. The volume of the whole wedge is of course $\frac{2}{3} AB^3$.

(A model for this figure would be useful)



Exercises

1. Show that the volume of the triangular pyramid in the figure is $\frac{1}{3}$ the area of the base times the height. (AB is perpendicular to BCD at B. No relation among the lengths AB, BC and BD is to be assumed).

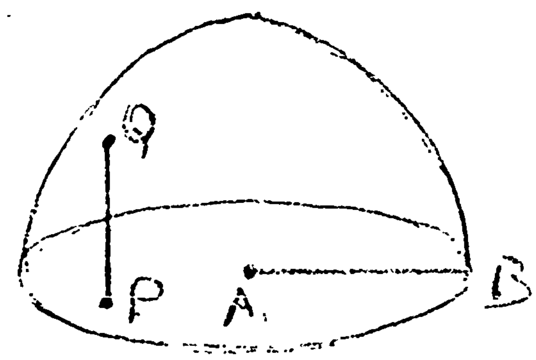
2. In the previous exercise, assume that the base remains the same but the point A moves in a plane parallel to BCD so that the height of the pyramid is unchanged. Will the volume be affected? Hint: Is there any change in the cross-sectional areas?

3. Apply the idea of Exercise 2 to show that the volume of any circular cone is $\frac{1}{3}$ the area of the base times the height.

Hint: See Section 6 but do not assume that the point A is necessarily above the center of the circle.

4. Two solids lie between a pair of parallel planes. Suppose that these solids are such that if they are cut by these planes and any third plane parallel to them, the area of the cross-section of the first solid is equal to the area of the corresponding cross-section of the second solid. Show that under these circumstances the two solids have the same volume. (This is called Cavalieri's Principle. Exercises 2 and 3 are special cases of it.)

8. Averages over Areas.



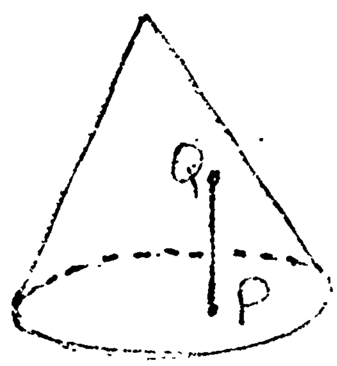
We now consider a new type of average. Suppose that we have a hemispherical solid of radius AB. Let P be any point in its base and PQ the height of a point Q on the hemispherical surface above the base. What is the average of PQ for all points on the base? We shall say that the average of PQ over the base is such that the volume is

$$\underline{PQ} \times \text{the area of the base.}$$

Then $\frac{2\pi}{3} AB^3 = \underline{PQ} \times \pi AB^2$

and $\underline{PQ} = \frac{2}{3} AB.$

If we had a cylinder with the same base as the hemisphere and the height $\frac{2}{3}$ AB, it would have exactly the same volume as the hemispherical region.

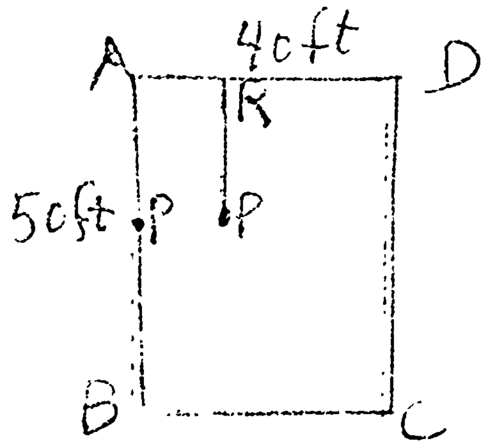


Similarly the average height ($\underline{PQ}$) of points Q on a conical surface above the base is one third the height of the cone.

This new type of average makes it possible to solve some interesting practical problems.

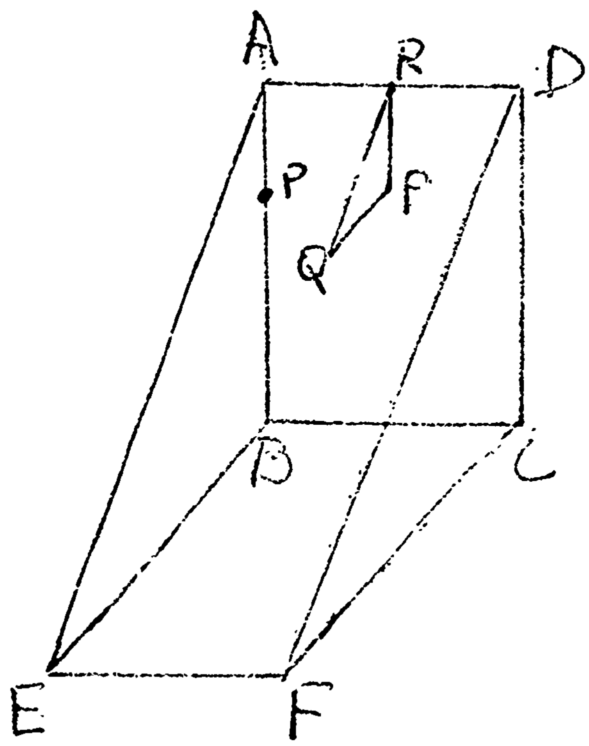
In Section 4 we found the force on a certain rectangular

dam. We did this by multiplying together 62.5 lbs per cu. ft., the average depth PA (=PR) and the total area.



Force in lbs. = 62.5 lbs./cu. ft. x PR x area ABCD.

It is possible to think of PR x area ABCD as the volume of a certain solid figure. Let P be any point on the dam at the depth PR below the surface. Draw a line of length PQ = PR perpendicular to the plane of ABCD. If this is done for all points P, a solid will be formed in the shape of a wedge whose volume is



$\frac{1}{2}$ BE x area ABCD

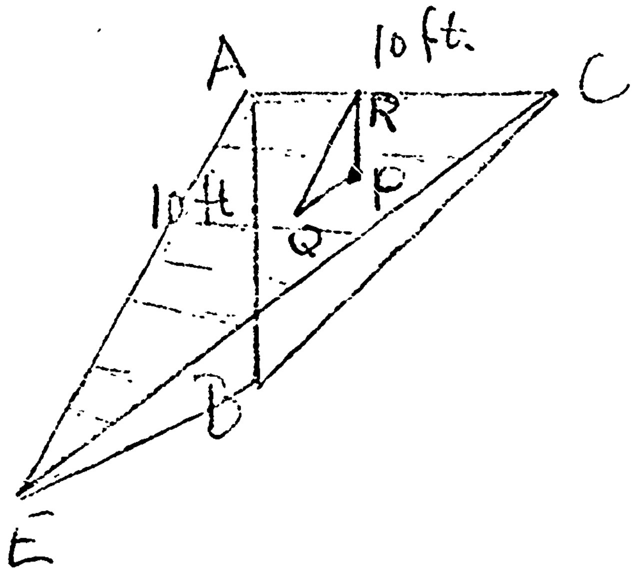
= $\frac{1}{2}$ AB x area ABCD

This is PQ x area ABCD

or PR x area ABCD.

We see therefore that the force in lbs. is 62.5 times the volume of the wedge ($\frac{1}{2}$ x 50 x 2000) in cu. ft.

Now suppose that we wish to find the force on the trianglar dam shown in the figure.

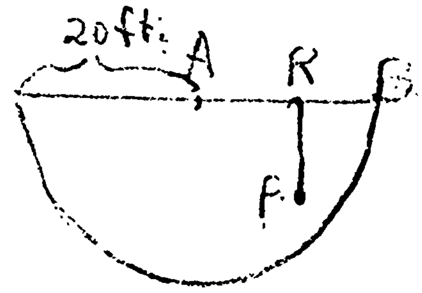


We know that the force in lbs. is 62.5 x $\underline{PR}$ x area ABC where $\underline{PR}$ is the average value of the depth PR for all points P of the triangular region. We proceed as before. At each point P draw a line segment $\overline{PQ}$ perpendicular to the plane of the triangle ABC so that PQ = PR. These perpendiculars fill up a triangular pyramid with base ABC and height BE = AB = 10. The volume of this pyramid is

$$\underline{PQ} \times \text{area } ABC$$
$$= \underline{PR} \times \text{area } ABC.$$

But the volume of course is $\frac{1}{3} \times 10 \times \frac{10 \times 10}{2} = \frac{500}{3}$ cu. ft. If we multiply by 62.5 lbs./cu. ft. we obtain the required force in lbs.

A similar problem is to find the force on a semicircular dam, say one of radius 20 ft. Again we must average the depth PR for all points of the semicircular region. If we erect a perpendicular $\overline{PQ}$ to the plane of the dam at each point P so that PQ = PR we obtain a solid which is the wedge whose volume we found in Section 6.



This volume is

$$\frac{2}{3}\ AB^3 = \frac{16000}{3} \text{cu. ft.}$$

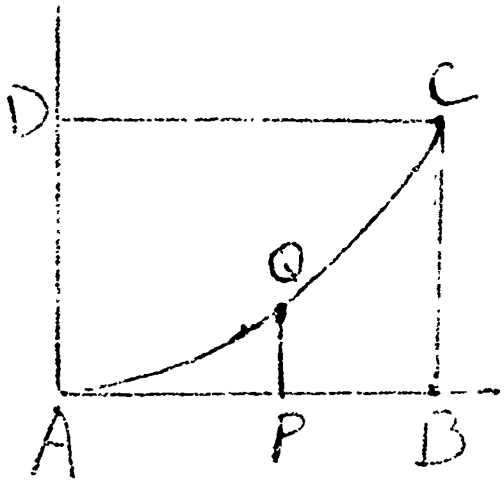
Since this volume is $\underline{PR}$ x the area of the semicircle, we have

$$\underline{PR} \times \text{area} = \frac{16000}{3}.$$

The total force is therefore

$$62.5 \times \frac{16000}{3} \text{ lbs.}$$

9. <u>A New Area Problem</u>



A curve is drawn in the following way. Above each point P on a line segment $\overline{AB}$ erect a perpendicular with length $PQ = AP^2$. For example if $AP = \frac{1}{2}\ AB$, PQ will be $\frac{1}{4}\ AB^2$ while if $AP = \frac{1}{3}\ AB$, PQ will be $\frac{1}{9}\ AB^2$. All points Q so constructed lie on a curve called a parabola. What is the area above $\overline{AB}$ and below the curve AC?

$$\text{Area} = \underline{PQ} \times AB = \underline{AP^2} \times AB = \frac{1}{3}\ AB^2 \times AB.$$

Since $BC = AB^2$ this result may be written

$$\text{Area} = \frac{1}{3}\ AB \times BC.$$

The area is seen to be $\frac{1}{3}$ the area of the rectangle ABCD.

This result was first obtained by Archimedes (about 250 B.C.) by a very difficult method. Most people think that it can be

obtained only by his method or by using calculus. As we see, the problem of finding the area under an arc of a parabola is equivalent to the problem of finding the volume of the simple square pyramid discussed in Section 5 and is therefore very elementary indeed.